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On Riesz and Cesàro Summability of Birkhoff Series

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On Riesz and Cesàro Summability of Birkhoff Series

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Abstract. For complex discrete Riesz means, as they occur in connection with eigenfunction expansions of Birkhoff type, regularity in the case of positive order, and equivalence to arithmetic means for order one are proven. Some applications to Birkhoff series are given.

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0. Introduction

In his well-known paper [13], M.H. Stone introduced Riesz means in order to derive summability results for eigenfunction expansions (Birkhoff series) associated with certain generally non-selfadjoint boundary value problems. He considered the expressions

$$-\frac{1}{2\pi i}\oint\limits_{|\lambda|=L_m}\int_0^1\left(1-\frac{\lambda^4}{L_m^4}\right)^\alpha G(x,\xi;\lambda)f(\xi)d\,\xi d\,\lambda, \tag{1}$$

where G is the Green's function of the corresponding boundary value problem, $\{L_m\}$ a sequence of positive values tending to infinity, and α a positive constant.† If $\lambda_1,\ldots,\lambda_k$ (with $|\lambda_1|\le|\lambda_2|\le\ldots\le|\lambda_k|$) are the poles, and $-g_1(x,\xi),\ldots,-g_k(x,\xi)$ the corresponding residues of G inside $|\lambda|=L_m$, then (1) can be written as

$$\sum_{i=1}^k\left(1-\frac{\lambda_i^4}{L_m^4}\right)^\alpha\int_0^1 g_i(x,\xi)f(\xi)d\,\xi \tag{2}$$

and thus can be viewed as a *complex discrete Riesz mean* for the partial sum

$$\sum_{i=1}^k\int_0^1 g_i(x,\xi)f(\xi)d\,\xi$$

of the eigenfunction expansion.

By means of his theorems V, IX, and XXXIII ([13], pp. 709, 721, and 747), Stone reduced the summation problem he considered to the Fourier case, whence he was able to use the classical theory of real Riesz means; for this theory see [6] and [1]. There remained a gap in Stone's argument which he filled in a second article [14].

In the present paper, we explicitly derive some properties of more general complex Riesz means

$$\sum_{\nu=0}^n\left(1-\frac{\lambda_\nu}{L_n}\right)^\alpha a_\nu, \tag{3}$$

since Stone's theorems and their generalizations (see [8] and [2], e.g.) do not apply to the case of uniform convergence and, moreover, do not lead to information on the regularity of Riesz means (3) and their relation

† For an introductory treatment of the basic concepts of the theory of Birkhoff series the reader may consult the monograph [9]. Stronger and more detailed expansion theorems are established, e.g., in the studies [2], [17], [7], [5], [12], [16], and [11].

to Cesàro means. In his Ph. D. thesis [7], F. J. Kaufmann proves a theorem on uniform Riesz summability which in view of the results of the present paper can be regarded as a generalization of Fejér's well-known theorem on uniform C_1 summability of the Fourier series development of a continous periodic function ([7], p.59ff.). In [5], p. 204ff., and [12], p. 135ff., analogous theorems for certain classes of indefinite boundary value problems are proven.

We hope that it will be of interest to workers in the field of summability to compare the present generalization of Riesz means to the different one studied in [4].

Our presentation makes use of notations and facts from [10].

1. A regularity theorem.

For a complex sequence $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ and a positive sequence $L = \{L_n\}_{n=0}^{\infty}$ we define the matrix $A_{L,\lambda}^{\alpha} = (a_{n\nu})$ with $\alpha > 0$ by

$$(4) \qquad \begin{cases} a_{00} := 1, \quad a_{0\nu} := 0 \quad (\nu > 0), \\ \text{and for } n \geq 1 \\ a_{n\nu} := \left(1 - \frac{\lambda_\nu}{L_n}\right)^{\alpha} - \left(1 - \frac{\lambda_{\nu+1}}{L_n}\right)^{\alpha} \quad (0 \leq \nu < n), \\ a_{nn} := \left(1 - \frac{\lambda_n}{L_n}\right)^{\alpha}, \\ a_{n\nu} := 0 \quad (\nu > n). \end{cases}$$

(The branch of the complex power is understood to be chosen in such a way that $\left(1 - \frac{\lambda}{L_n}\right)^{\alpha} = 1$ for $\lambda = 0$.)

Then a series $\sum a_n$ is called *summable* (L, λ, α) if the sums (3) converge to a finite limit for $n \to \infty$; equivalent: the sequence $s = \{s_n\}_{n=0}^{\infty}$ with $s_0 = a_0$, $s_n - s_{n-1} = a_n$ is *limitable* (L, λ, α), i.e. $\sum_{\nu=0}^{\infty} a_{n\nu} s_\nu = \sum_{\nu=0}^{n} a_{n\nu} s_\nu$ converges for $n \to \infty$.

Theorem 1.

Let $\lambda = \{\lambda_n\}$ be a sequence such that

$$(5a) \qquad \begin{cases} \lambda_0 = 0, \\ |\lambda_n| \to \infty \quad (n \to \infty), \\ |\lambda_{n+1}| > |\lambda_n| \quad \textit{for almost all } n \in \mathbf{N}, \end{cases}$$

and

$$(5b) \qquad |\operatorname{Im} \lambda_{n+1} - \operatorname{Im} \lambda_n| < c(\operatorname{Re} \lambda_{n+1} - \operatorname{Re} \lambda_n) \quad \textit{for almost all } n \in \mathbf{N}$$

with some positive constant c which is assumed to be < 1 for $\alpha < 1$. Then in case $|\lambda_n| < L_n \leq |\lambda_{n+1}|$ $(n \geq n_0)$ the $A_{L,\lambda}^{\alpha}$ method is regular (every convergent sequence is limitable (L, λ, α) to its limit) for all $\alpha > 0$.

Proof:

Two of the three well-known Toeplitz conditions (see [10], p. 11 f.) are obvious:

$$a_{n\nu} \to 0 \quad (n \to \infty) \quad (\text{“column conditon”}), \text{ since } |\lambda_n| \to \infty \quad \text{and} \quad L_n > |\lambda_n|;$$
$$\sum_{\nu=0}^{\infty} a_{n\nu} = \sum_{\nu=0}^{n} a_{n\nu} = 1 \ (\text{“row sum condition”}), \quad \text{since } \lambda_0 = 0.$$

We only have to prove the “row norm condition”

$$\sum_{\nu=0}^{\infty} |a_{n\nu}| = O(1) \quad (n \to \infty).$$

If $\alpha \geq 1$, we write for $\nu < n$

$$|a_{n\nu}| = \left| \left(1 - \frac{\lambda_\nu}{L_n}\right)^\alpha - \left(1 - \frac{\lambda_{\nu+1}}{L_n}\right)^\alpha \right|$$

$$= \frac{\alpha}{L_n} \left| \int_{\lambda_\nu}^{\lambda_{\nu+1}} \left(1 - \frac{\lambda}{L_n}\right)^{\alpha-1} d\,\lambda \right|$$

$$\leq \frac{\alpha 2^{\alpha-1}}{L_n} |\lambda_{\nu+1} - \lambda_\nu|,$$

hence

$$\sum_{\nu=0}^{\infty} |a_{n\nu}| \leq \alpha 2^{\alpha-1} \frac{1}{L_n} \sum_{\nu=1}^{n} |\lambda_\nu - \lambda_{\nu-1}| + 2^\alpha.$$

Now from (5b) we infer that for some $c_1 > 0$ and some n_1

$$|\lambda_{n+1} - \lambda_n| \leq c_1(\operatorname{Re}\lambda_{n+1} - \operatorname{Re}\lambda_n) \quad (n \geq n_1),$$

giving

$$\frac{1}{L_n} \sum_{\nu=1}^{n} |\lambda_\nu - \lambda_{\nu-1}| = o(1) + O\left(\frac{\operatorname{Re}\lambda_n}{L_n}\right) = O(1) \quad (n \to \infty).$$

If $0 < \alpha < 1$, we have for $\nu < n$

$$|a_{n\nu}| \leq \frac{\alpha}{L_n} \int_{\lambda_\nu}^{\lambda_{\nu+1}} \left(1 - \frac{|\lambda|}{L_n}\right)^{\alpha-1} |d\,\lambda|,$$

since $|1 - |z|| \leq |1 - z|$.
From (5b) (with $c < 1$ being essential) we conclude $|d\lambda| = O(d|\lambda|)$ $(\nu \to \infty)$ on the line segments joining λ_ν and $\lambda_{\nu+1}$, whence for some constant c_2 and $n_2 \leq \nu < n$

$$|a_{n\nu}| \leq c_2 \int_{|\lambda_\nu|}^{|\lambda_{\nu+1}|} \frac{\alpha}{L_n} \left(1 - \frac{|\lambda|}{L_n}\right)^{\alpha-1} d\,|\lambda|$$

$$= c_2 \left\{ \left(1 - \frac{|\lambda_\nu|}{L_n}\right)^\alpha - \left(1 - \frac{|\lambda_{\nu+1}|}{L_n}\right)^\alpha \right\}.$$

The row norm condition obviously follows. ■

2. Comparison with Cesàro means.

Now we restrict our attention to the case $\alpha = 1$ and impose additional conditions on the sequence λ in order to compare the Riesz means with arithmetic means. As might be expected, certain uniformity properties are required of the distances of consecutive λ_n.
From now on we assume

$$\text{(5c)} \qquad \frac{1}{\lambda_{n+1} - \lambda_n} - \frac{1}{\lambda_n - \lambda_{n-1}} = O\left(\frac{1}{\lambda_n}\right) \quad (n \to \infty).$$

In order to get a feeling for what this technical condition means we discuss some simple special cases.
Obviously

$$\frac{1}{\lambda_{n+1} - \lambda_n} - \frac{1}{\lambda_n - \lambda_{n-1}} = 0 \quad (n \in \mathbf{N})$$

implies $\lambda_n = n \cdot \lambda_1$, where $\operatorname{Re}\lambda_1 > 0$ because of (5b).
Another simple example is provided by $\lambda_n = n^\alpha$ with $\alpha > 0$:

$$\frac{1}{(n+1)^\alpha - n^\alpha} - \frac{1}{n^\alpha - (n-1)^\alpha} = \frac{\frac{1-\alpha}{\alpha} + O(\frac{1}{n})}{n^\alpha} \quad (n \to \infty). \tag{6}$$

Moreover, it is not difficult to prove some sort of converse of (6):
If $\{\lambda_n\}$ is a real sequence such that $\lambda_n \to \infty$ and

$$\frac{1}{\lambda_{n+1} - \lambda_n} - \frac{1}{\lambda_n - \lambda_{n-1}} = \frac{c + o(1)}{\lambda_n} \quad (n \to \infty) \tag{7}$$

holds for some $c > -1$, then for every $\varepsilon > 0$ there is an N such that

$$n^{\frac{1-\varepsilon}{1+c}} < \lambda_n < n^{\frac{1+\varepsilon}{1+c}} \quad (n \geq N).$$

Finally we mention that (5c) implies

$$\frac{1}{|\lambda_{n+1} - \lambda_n|} = O\left(\sum_{k=1}^{n} \frac{1}{|\lambda_k|}\right),$$

which in view of (5b) can also be written as

$$\frac{1}{\lambda_{n+1} - \lambda_n} = O\left(\sum_{k=1}^{n} \frac{1}{\lambda_k}\right). \tag{8}$$

Now we formulate conditions under which the $A^1_{L,\lambda}$ method can be compared with the Cesàro method.

Theorem 2.
Assume (5a,b,c). Then the C_1 method is not stronger than the $A^1_{L,\lambda}$ method (i.e. the $A^1_{L,\lambda}C_1^{-1}$ method is regular) for $|\lambda_n| < L_n \leq |\lambda_{n+1}|$ $(n \geq n_0)$ if and only if

$$\operatorname{Im}\lambda_n = O\left(\frac{1}{n}\lambda_n\right) \quad (n \to \infty), \tag{5d}$$

$$\lambda_{n+1} - \lambda_n = O\left(\frac{1}{n}\lambda_n\right) \quad (n \to \infty). \tag{5e}$$

Proof:
Since for both methods every row sum equals 1, whence this is true for $A^1_{L,\lambda}C_1^{-1}$, too, and since the column condition is trivial, we simply have to prove ([10], p. 16)

$$\sum_{\nu=0}^{n} (\nu+1)|a_{n\nu} - a_{n,\nu+1}| = O(1) \quad (n \to \infty), \tag{9}$$

in order to show the sufficiency of the conditions (5d,e). This relation in turn implies $(n+1)a_{nn} = O(1)$ and thus

$$1 - \frac{\lambda_n}{L_n} = O\left(\frac{1}{n}\right),$$

which for $L_n = |\lambda_{n+1}|$ leads to

$$\lambda_n = |\lambda_{n+1}|\left(1 + O\left(\frac{1}{n}\right)\right) = |\lambda_n|\left(1 + O\left(\frac{1}{n}\right)\right).$$

Hence the *necessity* of (5d,e) is obvious.
Now in order to derive (9) we use (5c), (5e) to get

$$\begin{aligned}\sum_{\nu=0}^{n-1}(\nu+1)|a_{n\nu}-a_{n,\nu+1}| &= \frac{1}{L_n}\sum_{\nu=0}^{n-1}(\nu+1)|(\lambda_{\nu+2}-\lambda_{\nu+1})-(\lambda_{\nu+1}-\lambda_\nu)| \\ &= \frac{1}{L_n}O\left(\sum_{\nu=0}^{n-1}\frac{\nu+1}{|\lambda_{\nu+1}|}|\lambda_{\nu+2}-\lambda_{\nu+1}|\cdot|\lambda_{\nu+1}-\lambda_\nu|\right) \\ &= \frac{1}{L_n}O\left(\sum_{\nu=1}^{n}|\lambda_\nu-\lambda_{\nu-1}|\right).\end{aligned}$$

By an argument already used in the proof of Theorem 1, and since $(n+1)|a_{nn}|=O(1)$ by (5d,e) (observe $|L_n-\lambda_n|\le|\lambda_{n+1}|-|\lambda_n|+O(\lambda_n/n)\le|\lambda_{n+1}-\lambda_n|+O(\lambda_n/n)=O(\lambda_n/n)$), we conclude (9). ■

We mention in passing that (5c) and (5e) together imply

(5c*) $$\lambda_{n+1}-\lambda_n=\left(1+O\left(\frac{1}{n}\right)\right)(\lambda_n-\lambda_{n-1}),$$

whereas (5c*) and (5e*) imply (5c); as is easily seen, (5c*) may replace (5c) in the proofs of the theorems 2, 3, and 4.

Theorem 3.

a) *If, in addition to* (5a,b,c),

(5e*) $$\frac{1}{n}\lambda_n=O(\lambda_{n+1}-\lambda_n)\quad(n\to\infty),$$

then for

$$\left|\frac{L_n-\lambda_{n+1}}{L_n-\lambda_n}\right|\le q<1\quad(n\ge n_0)$$

the $A^1_{L,\lambda}$ method is not stronger than the C_1 method.

b) *If, in addition to* (5a,b,c),

(5f) $$\liminf_{n\to\infty}\frac{|\lambda_{n+1}|}{|\lambda_n|}>1,$$

then for

$$L_n\le|\lambda_{n+1}| \text{ and } \left|\frac{L_n-\lambda_{n+1}}{L_n-\lambda_n}\right|\le 1\quad(n\ge n_0)$$

the $A^1_{L,\lambda}$ method is weaker than the C_1 method.

Proof:
First we compute the inverse $B=(b_{n\nu})$ of $A^1_{L,\lambda}$. If $\sigma_n=\sum_{\nu=0}^n a_{n\nu}s_\nu$ then $s_n=\sum_{\nu=0}^n b_{n\nu}\sigma_\nu$, and this leads to

$$b_{nn}=\frac{1}{a_{nn}},\quad b_{n\nu}=-\frac{1}{a_{\nu\nu}}\sum_{\mu=\nu+1}^{n}b_{n\mu}a_{\mu\nu}\quad(0\le\nu<n).$$

According to (4) we get

$$\frac{b_{nn}}{L_n}=\frac{1}{L_n-\lambda_n},\quad\frac{b_{n\nu}}{L_\nu}=\frac{\lambda_\nu-\lambda_{\nu+1}}{L_\nu-\lambda_\nu}\sum_{\mu=\nu+1}^{n}\frac{b_{n\mu}}{L_\mu}\quad(0\le\nu<n),$$

and with $(\lambda_\nu - \lambda_{\nu+1})/(L_\nu - \lambda_\nu) =: x_\nu - 1$ and thus

$$x_\nu := \frac{L_\nu - \lambda_{\nu+1}}{L_\nu - \lambda_\nu} \quad (0 \le \nu \le n)$$

we conclude

$$\text{(10)} \qquad \begin{cases} b_{nn} = \dfrac{(x_n - 1)L_n}{\lambda_n - \lambda_{n+1}}, \\ b_{n\nu} = \dfrac{(x_\nu - 1)L_\nu}{\lambda_n - \lambda_{n+1}} x_{\nu+1} \cdots x_{n-1}(x_n - 1), \quad 0 \le \nu < n. \end{cases}$$

Now we compute $D = (d_{n\nu}) := C_1\left(A^1_{L,\lambda}\right)^{-1}$.
Since

$$c_{n\nu} = \begin{cases} \frac{1}{n+1}, & 0 \le \nu \le n, \\ 0, & \nu > n, \end{cases}$$

we have

$$d_{n\nu} = \frac{1}{n+1}\sum_{\mu=\nu}^{n} b_{\mu\nu} = \frac{(x_\nu - 1)L_\nu}{n+1}\left\{\frac{1}{\lambda_\nu - \lambda_{\nu+1}} + \sum_{\mu=\nu+1}^{n} x_{\nu+1}\cdots x_{\mu-1}(x_\mu - 1)\cdot\frac{1}{\lambda_\mu - \lambda_{\mu+1}}\right\}.$$

Writing

$$a^{(\nu)}_\mu := \begin{cases} 1, & \mu = \nu, \\ x_{\nu+1}\cdots x_{\mu-1}(x_\mu - 1), & \mu > \nu, \end{cases}$$

whence

$$A^{(\nu)}_\mu := a^{(\nu)}_\nu + \ldots + a^{(\nu)}_\mu = x_{\nu+1}\cdots x_\mu \quad (\mu \ge \nu),$$

a summation by parts results in

$$\text{(11)} \qquad d_{n\nu} = \frac{(x_\nu - 1)L_\nu}{n+1}\left\{A^{(\nu)}_n \frac{1}{\lambda_n - \lambda_{n+1}} + \sum_{\mu=\nu}^{n-1} A^{(\nu)}_\mu\left(\frac{1}{\lambda_\mu - \lambda_{\mu+1}} - \frac{1}{\lambda_{\mu+1} - \lambda_{\mu+2}}\right)\right\}.$$

In order to show the regularity of the D method, we first observe that every row sum equals 1 by construction.
It remains to verify the column and row norm conditions.
From (11), (5c), (5e*) we conclude

$$|d_{n\nu}| = |x_\nu - 1| \cdot L_\nu \cdot O\left(\frac{|x_{\nu+1}\cdots x_n|}{|\lambda_{n+1}|} + \frac{1}{n+1}\sum_{\mu=\nu}^{n-1}\frac{|x_{\nu+1}\cdots x_\mu|}{|\lambda_{\mu+1}|}\right).$$

From $|x_n| \le 1$ $(n \ge n_0)$ the column condition immediately follows.
Now the estimate for $|d_{n\nu}|$ implies

$$\text{(12)} \qquad \sum_{\nu=0}^{n}|d_{n\nu}| = O\left(\sum_{\nu=0}^{n}\frac{L_\nu}{|\lambda_{n+1}|}|x_{\nu+1}\cdots x_n| + \frac{1}{n+1}\sum_{\mu=0}^{n-1}\sum_{\nu=0}^{\mu}\frac{L_\nu}{|\lambda_{\mu+1}|}|x_{\nu+1}\cdots x_\mu|\right).$$

Hence it suffices to show

$$\sum_{\nu=0}^{n}\frac{L_\nu}{|\lambda_{n+1}|}|x_{\nu+1}\cdots x_n| = O(1) \quad (n \to \infty).$$

In case $|x_\nu| \leq q < 1$ $(n \geq n_0)$ this is trivial (observe that $|(L_n - \lambda_{n+1})/(L_n - \lambda_n)| \leq q < 1$ implies $|L_n/\lambda_{n+1}| \leq (1+q)/(1-q)$).
Hence part a) of the theorem is proven.

If the assumptions of part b) hold, then $\lambda_{n+1} = O(\lambda_{n+1} - \lambda_n)$ whence (12) can be inferred in this case, too. From $|x_\nu| \leq 1$ and $L_\nu \leq O\big(q^{n-\nu}|\lambda_n|\big)$ $(0 \leq \nu \leq n)$ for some $q \in (0,1)$ we conclude

$$\frac{1}{|\lambda_{n+1}|} \sum_{\nu=0}^{n} L_\nu |x_{\nu+1} \cdots x_n| = O\left(\frac{1}{1-q}\right).$$

Thus the $A^1_{L,\lambda}$ method is not stronger than the C_1 method.
On the other hand, for almost all n

$$\frac{|L_n - \lambda_n|}{L_n} \geq \frac{|\lambda_{n+1} - \lambda_n|}{2L_n} \geq \frac{1}{2}\left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right|.$$

This contradicts (9), whence part b) follows. ■

As a corollary of the theorems 2 and 3 we get the following *equivalence theorem*.

Theorem 4 (Equivalence Theorem).
If, in addition to (5a,b,c,d),

$$(5\mathrm{e}^{**}) \qquad \begin{cases} \dfrac{c}{n}|\lambda_n| \leq |\lambda_{n+1} - \lambda_n| \leq \dfrac{C}{n}|\lambda_n| \quad (n \geq n_0) \\ \text{for some constants } c,\ C > 0, \end{cases}$$

then for

$$(13) \qquad |\lambda_n| < L_n \leq |\lambda_{n+1}| \quad \text{and} \quad \left|\frac{L_n - \lambda_{n+1}}{L_n - \lambda_n}\right| \leq q < 1 \quad (n \geq n_0)$$

a series is summable by the corresponding $A^1_{L,\lambda}$ method if and only if it is summable by arithmetic means.

It is not completely obvious that sequences $\{L_n\}$ obeying (13) always exist under the assumptions of Theorem 4. On the other hand, as is geometrically evident,

$$\operatorname{Im}(\lambda_{n+1} - \lambda_n) = o(\operatorname{Re}(\lambda_{n+1} - \lambda_n)) \ (n \to \infty),$$

$$\operatorname{Im}\lambda_n = O(\operatorname{Re}(\lambda_{n+1} - \lambda_n)) \ (n \to \infty)$$

together imply that there is a $q_0 \in (0,1)$ such that for sufficiently large n all Apollonius circles $|(z-\lambda_{n+1})/(z-\lambda_n)| = q$ with $q_0 \leq q < 1$ intersect the segment $(|\lambda_n|, |\lambda_{n+1}|)$ of the real axis.
Therefore under the hypotheses of Theorem 4 the general existence of sequences $\{L_n\}$ fulfilling (13) is an immediate consequence of the following fact.

Lemma 1.
The conditions (5b,c,d,e**) *together imply*

$$(5\mathrm{b}^*) \qquad \operatorname{Im}(\lambda_{n+1} - \lambda_n) = O\left(\frac{1}{\sqrt{n}}\operatorname{Re}(\lambda_{n+1} - \lambda_n)\right) \ (n \to \infty).$$

Proof:
First from (5b), (5d), and (5e**) we infer the existence of a constant $M > 0$ such that

$$(14) \qquad |\operatorname{Im}\lambda_n| \leq M \cdot \operatorname{Re}(\lambda_{n+1} - \lambda_n) \text{ for almost all } n.$$

Moreover, as was mentioned earlier, (5c) and (5e**) imply (5c*). Hence in view of (5b) there is a $d \geq 1$ such that for $n \geq n_0$ we have

$$\begin{aligned} \operatorname{Re}(\lambda_{n+1} - \lambda_n) &= (1 + \delta_n) \operatorname{Re}(\lambda_n - \lambda_{n-1}), \\ \operatorname{Im}(\lambda_{n+1} - \lambda_n) &= \operatorname{Im}(\lambda_n - \lambda_{n-1}) + \tilde{\delta}_n \operatorname{Re}(\lambda_n - \lambda_{n-1}) \end{aligned}$$

with $|\delta_n|, |\tilde{\delta}_n| < d/n$.
Assuming

$$|\operatorname{Im}(\lambda_{n+1} - \lambda_n)| \geq \varepsilon \operatorname{Re}(\lambda_{n+1} - \lambda_n)$$

with an $\varepsilon > 0$ for some $n \geq n_0$, we thus conclude (observe $1 + dx \leq (1+x)^d$)

$$\operatorname{Re}(\lambda_{n+j+1} - \lambda_{n+j}) \leq \prod_{i=1}^{j} \left(1 + \frac{d}{n+i}\right) \operatorname{Re}(\lambda_{n+1} - \lambda_n) \leq \left(\frac{n+j+1}{n+1}\right)^d \operatorname{Re}(\lambda_{n+1} - \lambda_n) \ (j \geq 0),$$

$$\begin{aligned} \operatorname{Im}(\lambda_{n+j+1} - \lambda_{n+j}) &\geq \left(\varepsilon - \frac{d}{n+1} - \frac{d}{n+2}\left(1 + \frac{d}{n+1}\right) - \cdots \right. \\ &\qquad \left. - \frac{d}{n+j}\left(1 + \frac{d}{n+1}\right) \cdots \left(1 + \frac{d}{n+j-1}\right)\right) \operatorname{Re}(\lambda_{n+1} - \lambda_n) \\ &= \left(\varepsilon - \left(\prod_{i=1}^{j}\left(1 + \frac{d}{n+i}\right) - 1\right)\right) \operatorname{Re}(\lambda_{n+1} - \lambda_n) \\ &\geq \left(\varepsilon - \left(\left(1 + \frac{j}{n+1}\right)^d - 1\right)\right) \operatorname{Re}(\lambda_{n+1} - \lambda_n) \ (j \geq 0) \end{aligned}$$

for $\operatorname{Im}(\lambda_{n+1} - \lambda_n) > 0$ (the case < 0 being analogous), and therefore (observe $dj/(n+1) \leq (1+j/(n+1))^d - 1 \leq d\, 2^{d-1} j/(n+1)$ for $0 \leq j \leq n+1$)

$$\begin{aligned} |\operatorname{Im}(\lambda_{n+k} - \lambda_n)| &\geq \left(k\varepsilon - \sum_{j=0}^{k-1}\left(\left(1 + \frac{j}{n+1}\right)^d - 1\right)\right) \operatorname{Re}(\lambda_{n+1} - \lambda_n) \\ &\geq k\left(\varepsilon - \frac{(k-1) d\, 2^{d-1}}{2n+2}\right) \operatorname{Re}(\lambda_{n+1} - \lambda_n) \ (0 \leq k \leq n+2). \end{aligned}$$

Since we may choose $k \approx n\varepsilon / d\, 2^{d-1}$ ($< n$ for sufficiently small ε), we infer $\varepsilon = O(1/\sqrt{n})$ from (14), whence the conclusion follows. ■

As a by-product we get the following growth restriction on λ_n.

Lemma 2.
The conditions (5b,c,d,e**) *together imply*

$$a \cdot n^c \leq |\lambda_n| \leq b \cdot n^C \quad (n \geq n_0)$$

with some $a, b > 0$, where the constants c and C are those of (5e**).

Proof:
We only derive the lower estimate, the other one being more straightforward.
The left part of (5e**), (5d), and Lemma 1 together imply

$$\operatorname{Re} \lambda_{n+1} = \operatorname{Re} \lambda_n \cdot \left(1 + \frac{c}{n} + O\left(\frac{1}{n^{3/2}}\right)\right).$$

Obviously, for sufficiently large n_0 there is a positive constant k such that

$$\prod_{n=n_0}^{N} \frac{1+\frac{c}{n}+O\left(\frac{1}{n^{3/2}}\right)}{1+\frac{c}{n}} \to k \quad (N \to \infty).$$

Hence for some $\tilde{k} > 0$

$$\operatorname{Re} \lambda_N \geq \tilde{k} \cdot \operatorname{Re} \lambda_{n_0} \cdot \prod_{n=n_0+1}^{N} \left(1+\frac{c}{n}\right).$$

The elementary estimates

$$(1+x)^c \leq 1+cx \leq (1+x)^c e^{cx^2/2} \quad (x \geq 0, 0 \leq c \leq 1)$$

and

$$(1+x)^c e^{-c^2x^2/2} \leq 1+cx \leq (1+x)^c \quad (x \geq 0, c \geq 1),$$

implying, e.g.,

$$e^{-c^2/2n} \left(\frac{n+j+1}{n+1}\right)^c \leq \prod_{i=1}^{j} \left(1+\frac{c}{n+i}\right) \leq \left(\frac{n+j+1}{n+1}\right)^c \quad (c \geq 1, j \in \mathbf{N}_0, n \in \mathbf{N}),$$

show that $\operatorname{Re} \lambda_n \geq$ const $\cdot n^c$ $(n \geq n_0)$, whence by (5d) the lower estimate of Lemma 2 follows. ∎

The lemmas show that (5a) — apart from the inessential condition $\lambda_0 = 0$ — is implied by (5b,c,d,e**).

3. Examples and application to Birkhoff series

When applying our theorems to concrete instances of $A^{\alpha}_{L,\lambda}$ methods the requirement $\lambda_0 = 0$ in (5a) is of no relevance (this condition being just a convenient standard assumption concerning the row sums); of course, changing finitely many L_n and λ_n does not affect the convergence properties of (3).

The first example we want to discuss is given by

$$\lambda_n = c \cdot n^{\alpha} \left(1+\frac{a}{n}+\frac{b}{n^{3/2}}+O\left(\frac{1}{n^2}\right)\right) \quad (n \to \infty) \tag{15}$$

with $c, \alpha > 0,\ a, b \in \mathbf{C}$.
Even from the less special representation

$$\lambda_n = c \cdot n^{\alpha} \left(1+\frac{a}{n}+o\left(\frac{1}{n}\right)\right)$$

we get

$$\operatorname{Re}(\lambda_{n+1}-\lambda_n) = \alpha c n^{\alpha-1} + o(n^{\alpha-1}),\ \operatorname{Im}(\lambda_{n+1}-\lambda_n) = o(n^{\alpha-1}),\ \left|\frac{\lambda_{n+1}}{\lambda_n}\right| = 1+\frac{\alpha}{n}+o\left(\frac{1}{n}\right),$$

whence (5a,b,d,e**) are obvious.
For (5c) we do need the more precise error term $O(n^{\alpha-2})$ implied by (15); from (15) we get

$$\Delta\lambda_n = \lambda_{n+1}-\lambda_n = \alpha c n^{\alpha-1} + O(n^{\alpha-2}),$$

$$\Delta^2\lambda_n = \Delta\lambda_{n+1}-\Delta\lambda_n = O(n^{\alpha-2}) = O\left(\frac{1}{n}\Delta\lambda_n\right),$$

which proves (5c*) and hence (5c), too. Thus according to Theorem 4 and Lemma 1 there are sequences $\{L_n\}$ such that the corresponding $A^1_{L,\lambda}$ method is equivalent to summation by arithmetic means.

Another instructive example is provided by

$$\lambda_n = c\cdot(\log n)^\alpha \left(1+\frac{a}{n}+\frac{b}{n^2}+O\left(\frac{1}{n^2\log n}\right)\right) \quad (n\to\infty) \tag{16}$$

with $c,\alpha>0,\ a,b\in\mathbf{C}$. We first prove (5c*). Using

$$\Delta n^\beta = \beta n^{\beta-1}+O(n^{\beta-2}),\ \Delta(\log n)^\beta = \beta\frac{(\log n)^{\beta-1}}{n}+O\left(\frac{(\log n)^{\beta-1}}{n^2}\right),$$

and

$$\Delta(a_nb_n) = (\Delta a_n)b_n + a_n(\Delta b_n) + (\Delta a_n)(\Delta b_n),$$

we conclude

$$\Delta\lambda_n = \alpha c\frac{(\log n)^{\alpha-1}}{n} - ac\frac{(\log n)^\alpha}{n^2}+O\left(\frac{(\log n)^{\alpha-1}}{n^2}\right),$$

$$\Delta^2\lambda_n = O\left(\frac{(\log n)^{\alpha-1}}{n^2}\right) = O\left(\frac{1}{n}\Delta\lambda_n\right).$$

(Observe that an error term $O\left((\log n)^\alpha/n^2\right)$ instead of $O\left((\log n)^{\alpha-1}/n^2\right)$ in (16) would be too large for (5c*) to hold.) From these computations we also infer

$$\mathrm{Re}(\lambda_{n+1}-\lambda_n) = \alpha c\frac{(\log n)^{\alpha-1}}{n}+O\left(\frac{(\log n)^\alpha}{n^2}\right),$$

$$\mathrm{Im}(\lambda_{n+1}-\lambda_n) = (\mathrm{Im}\,a)c\frac{(\log n)^\alpha}{n^2}+O\left(\frac{(\log n)^{\alpha-1}}{n^2}\right) = O\left(\frac{\log n}{n}\,\mathrm{Re}(\lambda_{n+1}-\lambda_n)\right),$$

$$\frac{\lambda_{n+1}}{\lambda_n} = 1+\frac{\Delta\lambda_n}{\lambda_n} = 1+\frac{\alpha}{n\log n}+O\left(\frac{1}{n^2}\right).$$

Hence (5a,b,c*,d,e) obviously hold, and since (5c*) may replace (5c) in the proof of Theorem 2, the $A^1_{L,\lambda}$ method is at least as strong as the C_1 method. A λ_n according to (16) *cannot* satisfy condition (5e*), since $\lambda_{n+1}-\lambda_n = \lambda_n(\alpha/n\log n + O(1/n^2))$, and it *need not* (and generally will not) satisfy condition (5c), as the "pure" case $\lambda_n = (\log n)^\alpha$ shows.

For

$$\lambda_n = q^n\,(1+o(1)) \quad (n\to\infty) \tag{17}$$

with $q>1$ the hypotheses of Theorem 3b) hold, and therefore the corresponding $A^1_{L,\lambda}$ method is weaker than the Cesàro method; nevertheless all $A^\alpha_{L,\lambda}$ methods are regular.

A more interesting example of non-equivalence:

$$\lambda_n = c\cdot n^\alpha\left(1+\frac{a+b\log n}{n}+O\left(\frac{1}{n^2}\right)\right) \quad (n\to\infty) \tag{18}$$

with $c,\alpha>0,\ a,b\in\mathbf{C},\ \mathrm{Im}\,b\neq 0$; in view of Theorem 2 the $A^1_{L,\lambda}$ method is *not* at least as strong as the Cesàro method in this case, since (5d) does not hold (whereas (5a,b,c,e**) are fulfilled).

We now want to apply our results to Birkhoff series. For this purpose we have to take into account that, as is well-known from the special case of classical Fourier series, the infinite linear combinations of eigenfunctions are computed by adding the summands *two* at a time. Thus in order to apply our equivalence theorem we need to slightly reformulate it.

Theorem 5 (Modified Equivalence Theorem).

*Let $\lambda' = \{\lambda'_n\}$ be a sequence for which (5a,b,c,d,e**) hold, and let $\lambda'' = \{\lambda''_n\}$ be a sequence such that*

$$\lambda'_n - \lambda''_n = O\left(\frac{1}{n}\lambda'_n\right) \ \ (n \to \infty). \tag{19}$$

Then, if $b_n \to 0$ $(n \to \infty)$, a series

$$\sum_{n \geq 0} (a_n + b_n)$$

is summable to the limit s by the method of arithmetic means if and only if it is summable to that limit by the modified Riesz method

$$\lim_{n\to\infty} \sum_{\nu=0}^{n} \left(\left(1 - \frac{\lambda'_\nu}{L_n}\right) a_\nu + \left(1 - \frac{\lambda''_\nu}{L_n}\right) b_\nu \right), \tag{20}$$

where $\{L_n\}$ is a sequence such that (13) holds with respect to the sequence λ'.

Proof:
In view of Theorem 4 we only have to show that

$$\sum_{\nu=0}^{n} \left(1 - \frac{\lambda'_\nu}{L_n}\right)(a_\nu + b_\nu) - \sum_{\nu=0}^{n} \left(\left(1 - \frac{\lambda'_\nu}{L_n}\right) a_\nu + \left(1 - \frac{\lambda''_\nu}{L_n}\right) b_\nu \right) = \sum_{\nu=0}^{n} \frac{\lambda''_\nu - \lambda'_\nu}{L_n} b_\nu$$

converges to zero for $n \to \infty$. Since $b_n \to 0$ for $n \to \infty$, this is a straightforward consequence of (19) and (5e**) ($\sum_{\nu=0}^{n} |(\lambda''_\nu - \lambda'_\nu)/L_n| = O(\sum_{\nu=1}^{n} |\lambda'_\nu - \lambda'_{\nu-1}|/L_n)$ is bounded for $n \to \infty$). ■

Since we want to address the case of *uniform* convergence of Birkhoff series, we have to add some observations concerning this point.

Lemma 3.

Let $A = (a_{n\nu})_{n,\nu=0}^{\infty}$ be a matrix fulfilling the Toeplitz conditions, and suppose that

$$b_n : D \to \mathbf{C} \ (n \in \mathbf{N}_0), \quad b_\infty : D \to \mathbf{C}$$

are functions such that b_∞ is bounded on D and

$$\lim_{n\to\infty} b_n(x) = b_\infty(x)$$

uniformly for $x \in D$.
Then

$$b_n^*(x) := \sum_{\nu=0}^{\infty} a_{n\nu} b_\nu(x) \quad (x \in D, n \in \mathbf{N}_0)$$

converges to $b_\infty(x)$ uniformly for $x \in D$, too.

Proof:
Use the estimate

$$|b_n^*(x) - b_\infty(x)| \leq \sum_{\nu=0}^{N} |a_{n\nu}| \cdot |b_\nu(x) - b_\infty(x)| + \sum_{\nu=N+1}^{\infty} |a_{n\nu}| \cdot |b_\nu(x) - b_\infty(x)|$$
$$+ |b_\infty(x)| \cdot \left| \sum_{\nu=0}^{\infty} a_{n\nu} - 1 \right|,$$

and apply the uniform convergence of $\{b_n\}$ as well as the Toeplitz conditions. ■

Now since Theorem 4 says that under the stated assumptions the $A^1_{L,\lambda}C_1^{-1}$ and $C_1(A^1_{L,\lambda})^{-1}$ methods are regular, in view of Lemma 3 we conclude that the equivalence statement extends to uniform convergence. Thus Theorem 5 also applies to the case of uniform convergence if we slightly amplify the assumptions made.

Theorem 6.

Under the assumptions of Theorem 5 a series

$$\sum_{n\geq 0}(a_n(x)+b_n(x))$$

is uniformly summable to a bounded function by the C_1 method if and only if it is uniformly summable to that function by the modified Riesz method (20), provided that the sequence $\{b_n(x)\}$ is uniformly bounded and converges to zero uniformly.

The generalization of Theorems 5 and 6 to the case of more than two summands is obvious.

Now the application of our theorems to Birkhoff series is straightforward. According to [9], p.64f., for a Birkhoff regular boundary value problem there are two sequences $\{\lambda'_n\}$ and $\{\lambda''_n\}$ of eigenvalues the asymptotic expansion of which generally leads to

$$(\lambda'_n)^4 = c\cdot n^\alpha\left(1+\frac{a'}{n}+O\left(\frac{1}{n^2}\right)\right), \tag{21a}$$

$$(\lambda''_n)^4 = c\cdot n^\alpha\left(1+\frac{a''}{n}+O\left(\frac{1}{n^2}\right)\right); \tag{21b}$$

there is a rare exceptional case (formula (47a,b) in [9], p.65), where we only get

$$(\lambda'_n)^4 = c\cdot n^\alpha\left(1+\frac{a}{n}+O\left(\frac{1}{n^{3/2}}\right)\right), \tag{22a}$$

$$(\lambda''_n)^4 = c\cdot n^\alpha\left(1+\frac{a}{n}+O\left(\frac{1}{n^{3/2}}\right)\right). \tag{22b}$$

In this special case we have to make slightly stronger smoothness assumptions concerning the coefficient functions of the differential operator in order to get the more precise asymptotic formula

$$c\cdot n^\alpha\left(1+\frac{a}{n}+\frac{b}{n^{3/2}}+O\left(\frac{1}{n^2}\right)\right) \tag{23}$$

for the eigenvalues.
The same formulas can be derived for the eigenvalues of regular boundary value problems in the indefinite case; this case is treated in [3], [5], and [12].
Thus the theorems on uniform Riesz summability in the case of a continuous function f fulfilling the (asymptotic) boundary conditions of order zero proved by Kaufmann and Stoeber (uniform convergence of the expansion (1) for $\alpha = 1$, see [7], pp.59–66, [12], pp.122–135,141) lead to theorems on uniform Cesàro summability via our general equivalence theorems. (Observe that for the expansions defined in [12], p.64f., condition (19) of Theorem 5 holds though there are several sequences of λ_n with different leading coefficients.) The elaborate direct equisummability proof given by Stoeber for the class of expansions he considered ([12], p.135–141) thus is only needed in the case of very weak smoothness assumptions concerning the coefficients of the differential operator; in most applications the smoothness assumptions required to apply our general summability theorems will be satisfied.

However, there is a case of local equisummability of Riesz and Cesàro means of expansions (1) which in view of Theorem 2 *cannot* be covered by our general theorems, since an asymptotic distribution of eigenvalues according to (18) is involved, the case of "Stone regular" or "extended regular" boundary value problems studied for the first time by Stone, and later in more general form by Eberhard, Freiling, Minkler, and others; see [15], [2], and [8].

4. Concluding remarks.

We hope that the generalized notion of Riesz summability presented in this paper, motivated by its relation to generalized Fourier series, will be of interest to some workers in the field of summability.
There remains a lot to be investigated, e.g. general theorems of consistency, and comparison theorems for Riesz and Cesàro means of orders other than one. In the case of order one, variations of the hypotheses used in our theorems deserve to be studied.

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